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Stability analysis of abnormal multiplication of plankton using parameter identification technique

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SUMMARY

This paper presents the mathematical approach for the abnormal multiplication of plankton. An abnormal multiplication can be expressed as an unstable problem and the stability of the system is investigated by introducing eigenvalues of a mathematical equation. The stability of the system can be judged by an eigenvalue based on the Lyapunov's stability theory. In this paper, the Arnoldi-QR method is used to obtain eigenvalues and eigenvectors of the system. The mode superposition method is employed to create spatial distribution needed to analyse the stability. To obtain the objective eigenvalue, the parameter identification technique is employed. The finite element method is used for the discretization in space. Lake Kasumigaura, which is located in Ibaraki Prefecture in Japan, is selected and actual data in 1975, 1976, 1991 and 2000 are used in order to investigate the stability of the specified lake in Japan. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: stability theory; finite element method; eigenvalue; parameter identification technique; mode superposition method

1. INTRODUCTION

In recent years, a lot of social problems have been occurring by the phenomenon of eutrophication in many lakes and marshes. For instance, red tide, a typical example shown in Figure 1 [1, 2], is one of the serious problems for the fishery industry, which is caused by the abnormal multiplication of plankton. This does not only destroy the balance of the ecosystem but also leads to the death of fish, seaweed, etc. Usually, the outbreak of plankton occurs in closed down water, such as lakes, marshes, bays, etc. as shown in Figure 2 [3, 4]. Recently, several approaches have been suggested to estimate the outbreak of plankton. However, it is very difficult to explain the actual observed data by the computational methods. In this research, an eigenvalue and eigenvector are introduced to judge the stability of the system considering the numerical calculation of the biological problems.

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Figure 1. Red tide.

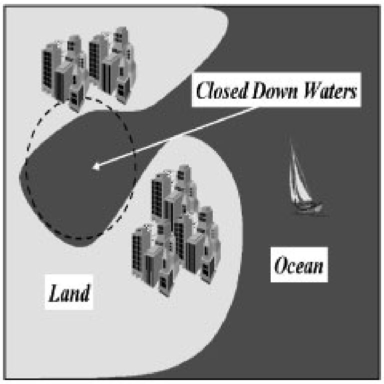


Figure 2. Closed down water.

It is known that the outbreak of plankton occurs suddenly. It is possible to think of this problem as the stability problem [5]. And in general, it is very difficult to estimate the multiplication of plankton and to research at which state the abnormal multiplication of plankton comes to the final stage. As the approach presented in this study, it is suggested to regard the phenomenon as the stability problem and to use the eigenvalue and eigenvector in order to judge the possibility that the system reaches the outbreak of plankton. Using this technique, it is possible to estimate when the water system would have abnormal multiplication of plankton in the future.

In this paper, abnormal multiplication of plankton is analysed as the two-dimensional biological model with diffusion to investigate the stability of the system. The system is sometimes unstable, but not always. Usually, the system is stable, but suddenly, it turns out to be unstable in space and in time. An eigenvalue problem is introduced based on the Lyapunov's stability theory. The system is judged as stable or unstable using the eigenvalue of the system, which can predict the abnormal multiplication of plankton.

To analyse the stability of the system, the spatial distribution should be calculated at the first stage. For this purpose, the mode superposition method is employed. The distribution is obtained by the superposition of the eigenmode of the domain and is made from the observation data obtained at the pre-assigned points. After obtaining the eigenvalue, optimal values of three parameters can be obtained with the parameter identification technique [6], where, the three parameters are introduced in the model equation and the parameters are multiplied to the equilibrium concentration of phytoplankton, zooplankton and nutrient. Using these parameters, it is possible to estimate the abnormal multiplication of plankton. The critical eigenvalue, with which the system is judged to be stable or not, can be obtained.

The finite element method is used for the discretization in space. As the case study, the Lake Kasumigaura, which is in Ibaraki Prefecture in Japan, is modelled in this research.

2. MATHEMATICAL MODEL

In this paper, the well-known mathematical model, presented by Wroblewski and O'Brien [9], is used, which can be expressed as

$$\dot{P} = D_1(P_{,xx} + P_{,yy}) + f(P,Z,N) \tag{1}$$

$$\dot{Z} = D_2(Z_{,xx} + Z_{,yy}) + g(P,Z,N) \tag{2}$$

$$\dot{N} = D_3(N_{,xx} + N_{,yy}) + h(P,Z,N) \tag{3}$$

where P, Z and N are the concentrations of phytoplankton, zooplankton and nutrient, respectively. In these equations, D_1, D_2 and D_3 are the non-dimensional diffusion coefficients of P, Z and N, respectively. The three functions $f(P,Z,N), g(P,Z,N)$ and $N(P,Z,N)$ are the

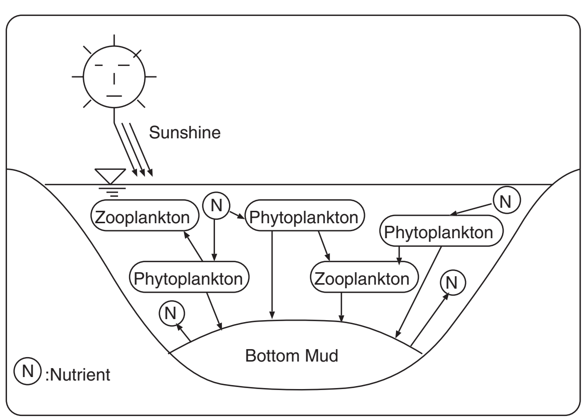


Figure 3. Food chain.

biological reaction terms, which are expressed in the following forms:

$$f(P,Z,N) = \frac{NP}{\alpha + N} - \beta Z[1 - \exp\{-\lambda(P - \hat{P})\}] - \psi P \tag{4}$$

$$g(P,Z,N) = \beta Z[1 - \exp\{-\lambda(P - \hat{P})\}] - \gamma\beta Z^2[1 - \exp\{-\lambda(P - \hat{P})\}] - \varphi Z \tag{5}$$

$$h(P,Z,N) = -\frac{NP}{\alpha + N} + \psi P + \gamma\beta Z^2[1 - \exp\{-\lambda(P - \hat{P})\}] + \varphi Z + \omega \tag{6}$$

The boundary condition is described as

$$P = \hat{P}, Z = \hat{Z}, N = \hat{N} \quad \text{on } \Omega_1 \tag{7}$$

$$\nabla P \cdot \mathbf{n} = \nabla Z \cdot \mathbf{n} = \nabla N \cdot \mathbf{n} = 0 \quad \text{on } \Omega_2 \tag{8}$$

where $\mathbf{n}$ is the unit outward normal vector, where the overhats in Equations (7) and (8) denote the prescribed value on the boundary.

3. STABILITY PROBLEM

3.1. Lyapunov's stability theory

The stability analysis is employed based on Lyapunov's stability theory. Following the theory, an equilibrium state and perturbation can be introduced to signify the stability of the system. Equilibrium state means a state to be examined and the perturbation is added as a disturbance. In the case where the system is completely stable, the disturbance settles down according to time. In the case where the system is unstable, the disturbance becomes large as the time passes. In this study, eigenvalue is used to judge if the system is stable or unstable. The judgement of the stability can be determined by the real part of eigenvalue σ of the system as

$$\mathrm{Re}\{\sigma\} < 0 : \text{Stable}$$

$$\mathrm{Re}\{\sigma\} = 0 : \text{Neutral}$$

$$\mathrm{Re}\{\sigma\} > 0 : \text{Unstable}$$

where Re{ } means the real part of the eigenvalue.

3.2. Linearization

In this research, the stability of a certain equilibrium state, which is the pre-assigned state, can be estimated. In order to obtain an eigenvalue of the system, the model equations are linearized following Lyapunov's Stability Theory. The equilibrium state is computed in a certain manner and perturbation is introduced to signify the stability. The eigenvalue can be obtained based on the linearized equation.

The following procedure is employed in order to linearize the equation. Consider the new solution $P_\tau + \Delta P, Z_\tau + \Delta Z$, and $N_\tau + \Delta N$, where the concentrations of phytoplankton, zooplankton and nutrient at the equilibrium state are denoted by P_τ, Z_τ, and N_τ. The disturbances $\Delta P, \Delta Z, \Delta N$ are brought to the equilibrium state which are assumed to be small. Substituting $P_\tau + \Delta P, Z_\tau + \Delta Z$, and $N_\tau + \Delta N$ into Equations (1)–(3), the following equations can be obtained:

$$\Delta \dot{P} = D_1 \nabla^2 (P_\tau + \Delta P) + f(P_\tau + \Delta P, Z_\tau + \Delta Z, N_\tau + \Delta N) \tag{9}$$

$$\Delta \dot{Z} = D_2 \nabla^2 (Z_\tau + \Delta Z) + g(P_\tau + \Delta P, Z_\tau + \Delta Z, N_\tau + \Delta N) \tag{10}$$

$$\Delta \dot{N} = D_3 \nabla^2 (N_\tau + \Delta N) + h(P_\tau + \Delta P, Z_\tau + \Delta Z, N_\tau + \Delta N) \tag{11}$$

Employing Taylor-expansion and omitting terms greater than the first order, the linearized equation of Equations (9)–(11) can be derived as

$$\Delta \dot{\phi} = F \cdot \Delta \phi \tag{12}$$

where

$$\Delta \phi = \begin{bmatrix} \Delta P \\ \Delta Z \\ \Delta N \end{bmatrix} \tag{13}$$

$$F = \begin{bmatrix} D_1 \nabla^2 + \dfrac{\partial f_\tau}{\partial P} & \dfrac{\partial f_\tau}{\partial Z} & \dfrac{\partial f_\tau}{\partial N} \\ \dfrac{\partial g_\tau}{\partial P} & D_2 \nabla^2 + \dfrac{\partial g_\tau}{\partial Z} & \dfrac{\partial g_\tau}{\partial N} \\ \dfrac{\partial h_\tau}{\partial P} & \dfrac{\partial h_\tau}{\partial Z} & D_3 \nabla^2 + \dfrac{\partial h_\tau}{\partial N} \end{bmatrix} \tag{14}$$

in which $\partial f_\tau / \partial P$ means the function that the values at the equilibrium state are substituted after differentiating with respect to P.

3.3. Allowance parameter λ

The estimation of how many concentrations at the pre-assigned state, i.e. the equilibrium state, have the allowance to the ones of the critical state are very important. To estimate the critical state, it is assumed that the critical state can be expressed by $\lambda_1 P_\tau + \Delta P, \lambda_2 Z_\tau + \Delta Z, \lambda_3 N_\tau + \Delta N$, where $\lambda(\lambda_1, \lambda_2, \lambda_3)$ is the allowance parameter and ΔP, ΔZ, ΔN are perturbations [6]. Equations (12)–(14) are transformed into

$$\Delta \dot{\phi} = F' \cdot \Delta \phi \tag{15}$$

$$\Delta\phi = \begin{bmatrix} \Delta P \\ \Delta Z \\ \Delta N \end{bmatrix} \tag{16}$$

$$F' = \begin{bmatrix} D_1\nabla^2 + \dfrac{\partial f_{\lambda\tau}}{\partial P} & \dfrac{\partial f_{\lambda\tau}}{\partial Z} & \dfrac{\partial f_{\lambda\tau}}{\partial N} \\ \Gamma \quad \dfrac{\partial g_{\lambda\tau}}{\partial P} & D_2\nabla^2 + \dfrac{\partial g_{\lambda\tau}}{\partial Z} & \dfrac{\partial g_{\lambda\tau}}{\partial N} \\ \dfrac{\partial h_{\lambda\tau}}{\partial P} & \dfrac{\partial h_{\lambda\tau}}{\partial Z} & D_3\nabla^2 + \dfrac{\partial h_{\lambda\tau}}{\partial N} \end{bmatrix} \tag{17}$$

Employing these parameters, the parameter identification technique can be used in this research and objective values of the allowance parameter will be obtained. The precise formulation will be presented in Section 5.

3.4. Finite element method

The perturbations are assumed in the following form:

$$\Delta P = P \cdot e^{\sigma t} \tag{18}$$

$$\Delta Z = Z \cdot e^{\sigma t} \tag{19}$$

$$\Delta N = N \cdot e^{\sigma t} \tag{20}$$

Introducing Equations (18)–(20) into Equation (15), the following equations can be obtained:

$$\sigma[M]\phi = [H]\phi \tag{21}$$

where σ is the eigenvalue used to estimate the critical state and

$$M = \begin{bmatrix} M_{\alpha\beta} & & \\ & M_{\alpha\beta} & \\ & & M_{\alpha\beta} \end{bmatrix} \tag{22}$$

$$H = \begin{bmatrix} -D_1 S_{\alpha\beta} + F_{P\alpha\beta} & F_{Z\alpha\beta} & F_{N\alpha\beta} \\ G_{P\alpha\beta} & -D_2 S_{\alpha\beta} + G_{Z\alpha\beta} & G_{N\alpha\beta} \\ H_{P\alpha\beta} & H_{Z\alpha\beta} & -D_3 S_{\alpha\beta} + H_{N\alpha\beta} \end{bmatrix} \tag{23}$$

in which

$$M_{\alpha\beta} = \int_V \Phi_\alpha \Phi_\beta \, \mathrm{d}V, \quad S_{\alpha\beta} = \int_V \Phi_{\alpha,i} \Phi_{\beta,i} \, \mathrm{d}V \tag{24}$$

and $F_{P\alpha\beta}$ is the discretized form of $\partial f_{\lambda\tau}/\partial P$.

4. EIGENVALUE PROBLEM

4.1. Arnoldi's method

To obtain the eigenvalue of the system, Arnoldi's method is applied in this research [6, 8]. This method enables to decrease the memory of dimension and computation time. The algorithm for the standard eigenvalues and eigenvectors problem ($Cu = \sigma u$) is as follows: 1: Start; Choose an initial vector v_1 of unity norm and a number of step m. 2: Iterate; For $j = 1, 2, \ldots, m$ do;

$$\hat{v}_{j+1} = Cv_j - \sum_{i=1}^{j} h_{ij} v_i \tag{25}$$

with

$$h_{ij} = (Cv_j, v_i), i = 1, \ldots, j \tag{26}$$

$$h_{j+1,j} = \|\hat{v}_{j+1}\|_2 \tag{27}$$

$$v_{j+1} = \hat{v}_{j+1}/h_{j+1,j} \tag{28}$$

This algorithm produces an orthonormal basis $V_m = [v_1, v_2, \ldots, v_m]$ of the Krylov subspace $K_m = \mathrm{span}\{v_1, Cv_1, \ldots, C^{m-1}v_1\}$. In this basis the restriction of C to K_m is represented by the upper Hessenberg matrix H_m, whose entries are the h_{ij} produced by the algorithm, i.e.

$$H_m = h_{ij}$$

The eigenvalues of C are approximated by those of H_m which is obtained as follows.

$$H_m = V_m^{\mathrm{T}} C V_m$$

where one wishes to choose m sufficiently small so that the work in generating H and computing its eigenvalues by the QR method is not excessive, but m needs to be sufficiently large so that the selected eigenvalues of C are approximated accurately.

4.2. Application for generalized eigenvalue problem

If one wishes to find out the leading eigenvalue with maximum real part, it is common to use the shift and invert strategy. If σ_0 is an approximation to an eigenvalue of interest, then the shifted and inverted problem is

$$(C - \sigma_0 I)^{-1} u = \mu u \tag{29}$$

where $\mu = 1/(\sigma - \sigma_0)$. Thus, eigenvalues of C close to σ_0 correspond to eigenvalues μ of Equation (29) with a large absolute value and one expects Arnoldi's method to converge to such eigenvalues.

In order to apply Arnoldi's method to Equation (29) for the generalized eigenvalue problem Equation (30), Equation (29) may be described as

$$(H - \sigma_0 M)^{-1} M u = \mu u \tag{30}$$

and to apply to Arnoldi's method, the LU decomposition of $H - \sigma_0 M$ is carried out once, and then each time $(H - \sigma_0 M)^{-1} M v$ is needed, $(H - \sigma_0 M)w = Mv$ is solved by forward and backward procedures. This is much more economical than forming the matrix of Equation (25) explicitly since it is usually full and also its dimension is much larger than m.

5. PARAMETER IDENTIFICATION

5.1. Performance function

In order to obtain the allowance parameters $(\lambda_1, \lambda_2, \lambda_3)$, a parameter identification technique is used in this research. If the parameters included in the model equations change, the stability of the system can also change. The critical parameter values change the state from stable to unstable or unstable to stable. The parameter identification technique can be usefully employed to determine the critical allowance parameters.

This technique is equal to the estimation with minimization of performance function J, which means the sum of squared residual between calculated and objective values. This function is described as

$$J = \frac{1}{2} \int_V (\mathrm{Re}\{\sigma(\lambda)\} - \sigma^*)^{\mathrm{T}} (\mathrm{Re}\{\sigma(\lambda)\} - \sigma^*)\, \mathrm{d}V \tag{31}$$

$$\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots, \sigma_n^*)^{\mathrm{T}}, \quad \mathrm{Re}\{\sigma(\lambda)\} = \mathrm{Re}\{(\sigma(\lambda)_1, \sigma(\lambda)_2, \sigma(\lambda)_3, \ldots, \sigma(\lambda)_n)^{\mathrm{T}}\}$$

where $\sigma(\lambda)$ is the eigenvalue of the system, n is the total number of nodal points, and σ^* are objective eigenvalues. The critical eigenvalue $\mathrm{Re}\{\sigma\} = 0$ is used as the objective value in this paper. However, it is negative and is defined in order to be nearly equal to zero, because the system changes from unstable to stable and the critical state can be obtained. The allowance parameter value can be solved to minimize the function J.

The Fletcher-Reeves method, which is one of the conjugate gradient methods, is used in this analysis. There are two advantages in this method, for example, the algorithm is very simple and a stable solution can be obtained. The search direction d at the first step is computed by the following equation:

$$\{d^{(0)}\} = -\left\{ \frac{\partial J}{\partial \lambda^{(0)}} \right\} = -\int \left[\frac{\partial \sigma}{\partial \lambda^{(0)}} \right]^{\mathrm{T}} (u - u^*)\, \mathrm{d}v \tag{32}$$

where $[\partial\sigma/\partial\lambda]$ is referred to as the sensitivity matrix. The boundary condition of the sensitivity matrix is written as

$$\left[\frac{\partial\sigma}{\partial\lambda}\right]=0 \quad \text{on } \Omega_1 \tag{33}$$

Considering the performance function $J(\lambda+\alpha d)$ using the gradient d

$$J(\{\lambda\}_i+\alpha\{d\})=\frac{1}{2}\int\left\{\sigma(\lambda_i)+\alpha\left[\frac{\partial\sigma}{\partial\lambda}\right]d_i-\sigma^*\right\}^{\mathrm{T}}\left\{\sigma(\lambda_i)+\alpha\left[\frac{\partial\sigma}{\partial\lambda}\right]d_i-\sigma^*\right\}\mathrm{d}v \tag{34}$$

the step size α that gives the minimum value of Equation (34) is obtained by partially differentiating function $J(\{\lambda\}_i+\alpha\{d\})$ with respect to α and setting the resultant equation equal to zero.

$$\alpha=-\frac{d_i^{\mathrm{T}}\left\{\dfrac{\partial J}{\partial\lambda}\right\}}{d_i^{\mathrm{T}}\left[\dfrac{\partial\sigma}{\partial\lambda}\right]^{\mathrm{T}}\left[\dfrac{\partial\sigma}{\partial\lambda}\right]d_i} \tag{35}$$

The parameter is renewed using $\{d\}$ and α, which are obtained by Equations (32) and (35), respectively. The new parameter is expressed as

$$\{d\}_{i+1}=-\left\{\frac{\partial J}{\partial\lambda}\right\}_{i+1}+\beta_i\{d\}_i \tag{36}$$

where

$$\beta=\frac{\left\{\dfrac{\partial J}{\partial\lambda}\right\}_{(i+1)}^{\mathrm{T}}\left\{\dfrac{\partial J}{\partial\lambda}\right\}_{(i+1)}}{\left\{\dfrac{\partial J}{\partial\lambda}\right\}_{(i)}^{\mathrm{T}}\left\{\dfrac{\partial J}{\partial\lambda}\right\}_{(i)}} \tag{37}$$

The renewed d_{i+1} is used for the search direction at the next iterative stage $i+1$.

5.2. Algorithm

In this research, the Fletcher Reeves method is employed to minimize J. The algorithm of the parameter identification technique is as follows:

1. Assume initial parameter value $\lambda^{(0)}$, choose convergence criterion ε_J
2. Calculate state value $\sigma(\lambda)^{(0)}$
3. Calculate performance function $J^{(0)}$
4. Calculate sensitivity matrix $[\partial\sigma(\lambda)/\partial\lambda]^{(0)}$
5. Calculate initial gradient $d^{(0)}=-\{\partial J/\partial\lambda\}^{(0)}$
6. Calculate step size α so as to minimize $J(\sigma^{(i)}+\alpha d^{(i)})$
7. Renew parameter $\lambda^{(i+1)}=\lambda^{(i)}+\alpha d^{(i)}$
8. Calculate state value $\sigma(\lambda)^{(i+1)}$
9. Calculate performance function $J^{(i+1)}$

10. Calculate sensitivity matrix $[\partial\sigma(\lambda)/\partial\lambda]^{(i+1)}$
11. Calculate $\beta = \{\partial J/\partial\lambda\}^{(i+1)}\{\partial J/\partial\lambda\}^{(i+1)}/\{\partial J/\partial\lambda\}^{(i)}\{\partial J/\partial\lambda\}^{(i)}$
12. Calculate gradient of performance function J and $d^{(i+1)} = -\{\partial J/\partial\lambda\}^{(i+1)} + \beta d^{(i)}$
13. If $|J^{(i+1)} - J^{(i)}| < \varepsilon_J$, then stop
14. Set $i = i+1$ and go to 6

5.3. Sensitivity matrix

In order to solve the sensitivity matrix of the eigenvalue, the left eigenvalue problem has to be used in this study, which is

$$\sigma A\phi + B\phi = 0 \tag{38}$$

$$\sigma A^{\mathrm{T}}\varphi + B^{\mathrm{T}}\varphi = 0 \tag{39}$$

where A^{T} (or B^{T}) is the transposed matrix of $A(B)$. The eigenvectors of Equations (38) and (39) are not the same, but the eigenvalues are the same. In this research, the maximum eigenvalue of the real part is investigated and eigenvectors of real and imaginary parts are used by solving the sensitivity matrix of the real part. The real part of the sensitivity matrix can be obtained in the following manner:

$$\mathrm{Re}\left\{\frac{\partial\sigma}{\partial\lambda}\right\} = -\,\mathrm{Re}\left\{\frac{\varphi^{\mathrm{T}}\left(\sigma\dfrac{\partial A}{\partial\lambda} + \dfrac{\partial B}{\partial\lambda}\right)\phi}{\varphi^{\mathrm{T}}A\phi}\right\} \tag{40}$$

By calculating those matrices, the parameter identification technique can be usefully applied. Finally, the optimal parameter value which can make the system stable can be obtained.

6. MODE SUPERPOSITION METHOD

In the analysis presented in this paper, the equilibrium state should be pre-assigned. For this purpose, the observation can be effectively used. However, observation data can be obtained at the points of the limited numbers. Therefore, for the computational purpose, it is necessary that the data of the spatial distribution of the pre-assigned state should be computed at all nodal points. To do this, the Mode Superposition Method is used.

6.1. Helmholtz equation

The superposition of the eigenmode of the Helmholtz equation is utilized in this study to carry out the spatial distribution. The Helmholtz equation is

$$\nabla^2\phi + k^2\phi = 0 \tag{41}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{42}$$

In Equation (41), ϕ presents the basic mode of biological data of phytoplankton, zooplankton and nutrient. The boundary condition is

$$\nabla\phi \cdot \mathbf{n} = 0 \tag{43}$$

The finite element method is used for the spatial discretization. To obtain the eigenvalues k^2, the Householder-QR Method is used and eigenvectors ϕ are computed by the backward substitution.

6.2. Eigenvalue problem by FEM

To obtain eigenvalues k^2 and eigenvectors ϕ, the finite element method is employed to discretize Equation (41):

$$S_{\alpha\beta}\phi_\beta - k^2 M_{\alpha\beta}\phi_\beta = 0 \tag{44}$$

where

$$S_{\alpha\beta} = \int_\Omega \Phi_{\alpha,i},\ \Phi_{\beta,i}\,\mathrm{d}\Omega, \qquad M_{\alpha\beta} = \int_\Omega \Phi_{\alpha i}\,\Phi_{\beta i}\,\mathrm{d}\Omega \tag{45}$$

Equation (44) is dealt with as the general eigenvalue problem. Matrix $M_{\alpha\beta}$ is symmetric, thus, the matrix can be decomposed into two matrices by the Choleski Method as

$$M_{\alpha\beta} = L_{\alpha\beta}^{\mathrm{T}} L_{\alpha\beta} \tag{46}$$

Substituting Equation (46) into Equation (44), then

$$S_{\alpha\beta}\phi_\beta - k^2 L_{\alpha\beta}^{\mathrm{T}} L_{\alpha\beta}\phi_\beta = 0 \tag{47}$$

Eigenvector ϕ is replaced by $u_\beta = L_{\alpha\beta}\phi_\beta$, thus

$$S_{\alpha\beta} L_{\alpha\beta}^{-1} u_\beta = k^2 L_{\alpha\beta}^{\mathrm{T}} L_{\alpha\beta} L_{\alpha\beta}^{-1} u_\beta \tag{48}$$

$$k^2 u_{\alpha\beta} = L_{\alpha\beta}^{-\mathrm{T}} S_{\alpha\beta} L_{\alpha\beta}^{-1} u_{\alpha\beta} \tag{49}$$

where

$$A_{\alpha\beta} = L_{\alpha\beta}^{-\mathrm{T}} S_{\alpha\beta} L_{\alpha\beta}^{-1} \tag{50}$$

substituting Equation (50) into Equation (49), the following equation can be obtained:

$$k^2 u_\beta = A_{\alpha\beta} u_\beta \tag{51}$$

To find the eigenvalues k^2 and the eigenvectors u_β from Equation (51), the Householder-QR Method and the Inverse Iteration Method is employed. The function ϕ is equal to $L^{-1}u$, the eigenvector ϕ is found by the Backward Substitution Method.

6.3. Superposition

The mode superposition method can be described as follows. One of the concentrations of phytoplankton, zooplankton and nutrient at the pre-assigned state is expressed by $\hat{u}$, which is considered as a state vector. The state vector can be assumed to be expressed by

$$\hat{u} = \sum_{i=1}^{n} u_i c_i \tag{52}$$

where u_i is the eigenmode of the Helmholtz equation and c_i is the unknown constant, which corresponds to the vector which should be determined. The observation data at the observation points are denoted by $\tilde{u}$, which are known constants. The performance function is

$$J = \sum_{\Omega} (\tilde{u} - \hat{u})^2 \tag{53}$$

The problem can be converted as follows. Find c_i so as to minimize $J(c_i)$. Namely, find the unknown c_i so as to minimize the distance between observed and computed data. The function J can be written as

$$J = \frac{1}{2} \sum_{j=1}^{mx} (\hat{u}_j - \tilde{u}_j)^2 \tag{54}$$

$$= \frac{1}{2} \sum_{j=1}^{mx} (\hat{u}_j^2 - 2\hat{u}_j \tilde{u}_j + \tilde{u}_j^2) \tag{55}$$

$$= \frac{1}{2} \sum_{j=1}^{mx} \left[\left(\sum_{i=1}^{n} u_{ij} c_i \right)^2 - 2 \left(\sum_{i=1}^{n} u_{ij} c_i \right) \tilde{u}_j + \tilde{u}_j^2 \right] \tag{56}$$

$$\tag{57}$$

using this, the derivatives of J can be computed.

$$\frac{\partial J}{\partial c_l} = \frac{1}{2} \sum_{j=1}^{mx} \left[\left(2 \sum_{i=1}^{n} u_{ij} c_i \right) u_{ij} - 2 u_{lj} \tilde{u}_j \right] \tag{58}$$

$$= \sum_{j=1}^{mx} \left[\left(\sum_{i=1}^{n} u_{ij} c_i \right) u_{lj} - u_{lj} \tilde{u}_j \right] \tag{59}$$

$$= \sum_{j=1}^{mx} \left[u_{lj} \left(\sum_{i=1}^{n} u_{ij} c_i - u_{lj} \right] \tag{60}$$

$$(l = 1, 2, 3, \ldots, n)$$

The conjugate gradient method is employed for the above equations to obtain the vector c_i to be determined. The algorithm of the Fletcher Reeves Method is expressed as follows:

1. Assume initial parameter $c^{(0)}$, decide convergence criterion ε_J, ε_c and set $i = 0$
2. Compute performance function $J^{(0)}$
3. Compute gradient of performance function; $d^{(0)} = -\{\partial J/\partial c\}^{(0)}$
4. Solve step width α to minimize $J(c^{(i)} + \alpha d^{(i)})$
5. Renovate parameter c; $c^{(i+1)} = c^{(i)} + \alpha d^{(i)}$
6. Compute $J^{(i+1)}$ and $d^{(i+1)}$
7. If $|J^{(i+1)} - J^{(i)}| < \varepsilon_J$, $\|c^{(i+1)} - c^{(i)}\| < \varepsilon_c$ then stop else go to 8
8. $i = i + 1$ and go to 4

The Fletcher Reeves method is employed in this research. This algorithm is simple and the computation is reasonably stable.

7. CASE STUDY

7.1. Lake Kasumigaura

Lake Kasumigaura is chosen to be analysed as the case study. This lake consists of three small lakes: Nishi-Ura, Kita-Ura and Soto-Nasaka Ura. The lakes are located in the southeast of Ibaraki Prefecture in Japan. The total area is 220 km^2, making them second in size, within Japan, to Biwa Ko in Shiga Prefecture. The catchment area of the lake is 2160 km^2, occupying approximately 35% of the land within Ibaraki Prefecture.

In this lake, there have been water quality problems and its damage has been very serious.

One of the well-known problems is the outbreak of 'Microcystis aeruginosa'. It is a kind of phytoplankton like red tide. By eutrophication, the water quality problem like 'Microcystis aeruginosa' occurred. The location of Lake Kasumigaura is shown in Figure 4. Figure 5 shows the mesh employed in this study.

It is considered in this study that the outbreak of Microcystis aeruginosa is related to the stability problem based on the eigenvalues. The basic equation, Equations (1)–(3), include various parameters. The parameter values are described in Table I. The values of biological

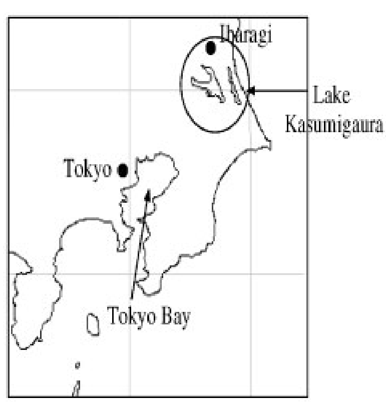


Figure 4. Lake Kasumigaura.

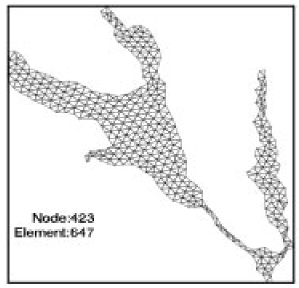


Figure 5. Finite element mesh.

Table I. Parameter values.

Parameter	Meaning	
α	Michaelis constant	0.1
β	Zooplankton maximum grazing rate	1.2
γ	Zooplankton egestion coefficient	2.31
λ	Ivlev constant	1.8
$\hat{P}$	Zooplankton grazing threshold	0.083
ϕ	Phytoplankton nutrient loss coefficient	0.15
φ	Zooplankton death rate	0.25
ω	Nutrient generation term from the inside	0.0005

parameters always change, and are not the same values, and these values are original in each lake. In this research, the data in Table I observed by the Fluid Pollution of Environment [9] are used, which are the average values observed from 1973 to 1976 in Lake Kasumigaura. The diffusion coefficients D_1, D_2 and D_3 in Reference [4] are also employed.

7.2. Spatial distribution

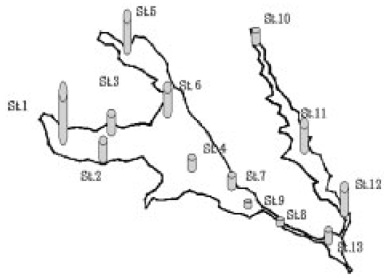
The observation at 13 points in the Lake Kasumigaura are carried out. In this research, to assign the equilibrium state, the spatial values of the observed data at all nodal points should be necessary. In order to create this equilibrium state, eigenvalues (1–20) of the Helmholtz equation (Equation (41)) are utilized and the superposing of eigenmode is employed for the mode superposition method. Using the mode superposition method, the distribution can be obtained based on the relatively small number of data.

Figure 6 shows the observation points employed in this research.

The results of the mode superposition method are shown in Figure 7. This is the spatial distribution of the pre-assigned state in May 1976. In order to create each distribution, the concentration data at 13 points are given. Table II shows the concentration at each point.

In Table II, Phy., Zoo., and Nut. indicate the concentration of Phytoplankton, Zooplankton, and Nutrient, respectively. Each value is non-dimensional because the mathematical model employed in this research are non-dimensional equations.

Three components (P, Z, N) are calculated with this method, and stability analysis is conducted based on these functions as the equilibrium state.



St.1:Kakema	St.8:Asou
St.2:Kihara	St.9:Itako
St.3:Ushigome	St.10:Takei
St.4:Koshin	St.11:Kamagaya
St.5:Takasaki	St.12:Jingubashi
St.6:Tamatukuri	St.13:Sotonami-Sakaura
St.7:Nishinosu	

Figure 6. Observation points.

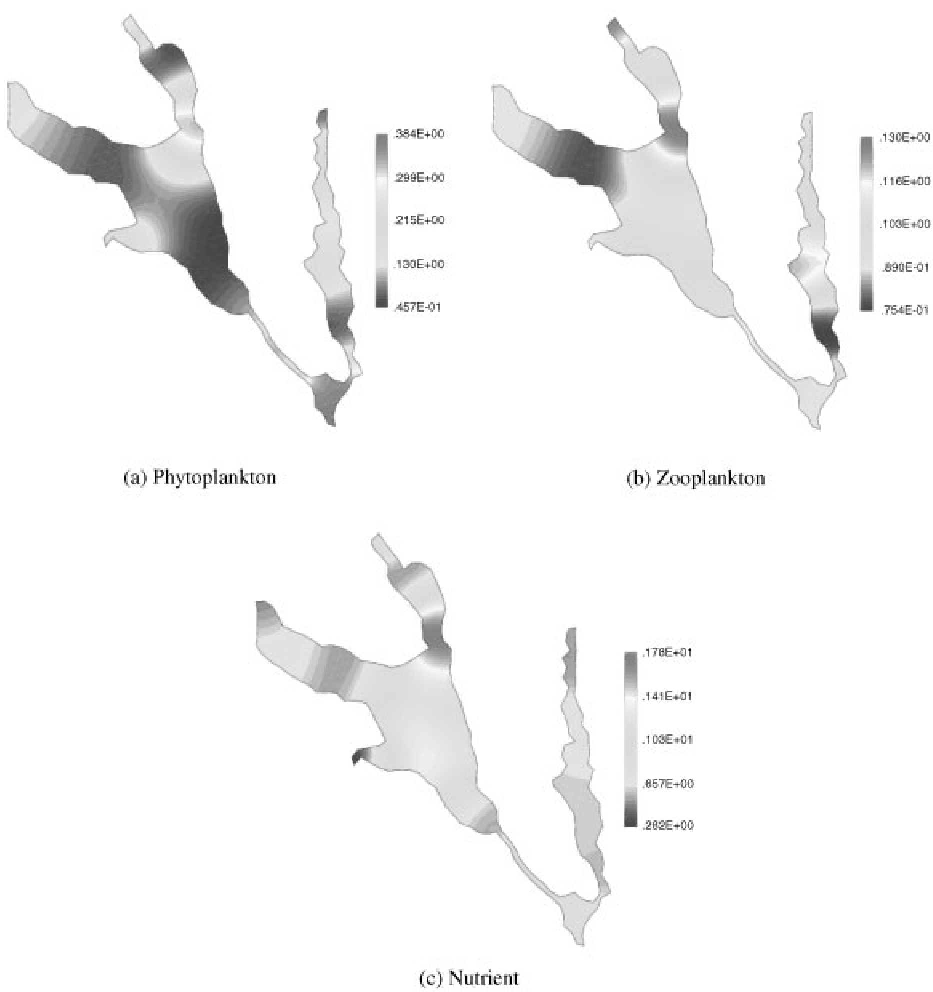


Figure 7. Concentration of P, Z and N: (a) phytoplankton, (b) zooplankton, and (c) nutrient.

Table II. Concentration data at 13 points.

	St.1	St.2	St.3	St.4	St.5	St.6	St.7	St.8	St.9	St.10	St.11	St.12	St.13
Phy.	0.14	0.09	0.08	0.09	0.32	0.10	0.07	0.09	0.12	0.37	0.18	0.18	0.22
Zoo.	0.09	0.09	0.08	0.10	0.13	0.09	0.10	0.11	0.11	0.11	0.10	0.12	0.11
Nut.	0.66	0.64	0.55	0.56	1.68	1.29	0.80	0.60	1.02	0.87	0.77	0.65	0.58

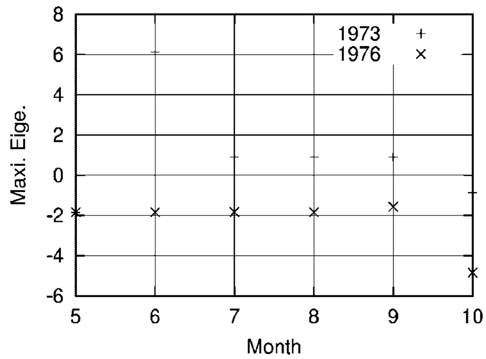


Figure 8. Maximum Eigenvalue.

Table III. Maximum eigenvalue of real part.

	May	Jun.	Jul.	Aug.	Sep.	Oct.
1973	−1.84628	6.12549	0.91376	0.90946	0.90455	−0.87963
1976	−1.83072	−1.83932	−1.82642	−1.83857	−1.56453	−4.83910

Multiplying 10^{-2} to each value.

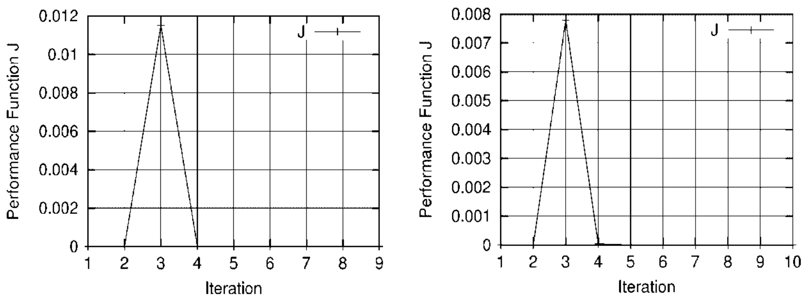
7.3. Numerical results

The maximum eigenvalue of the real part is investigated (see Figure 8). It is confirmed that the stability of the system using the eigenvalue is adaptable. If the eigenvalue is negative, the system is stable. If it is positive, the system is regarded to be unstable, which means that the outbreak of plankton has occurred. For example, by employing the parameter values in Table I and spatial distribution in Figure 7, the eigenvalues in 1973 and 1976 are calculated.

Table III shows computed results. In June, July, August and September in 1973, the eigenvalue is positive, which means that the system is unstable. Contrary to this, all data in 1976 and in May and October in 1973 are negative, which means the system is stable. In the summer in 1973, actual outbreak of plankton was observed in the lake.

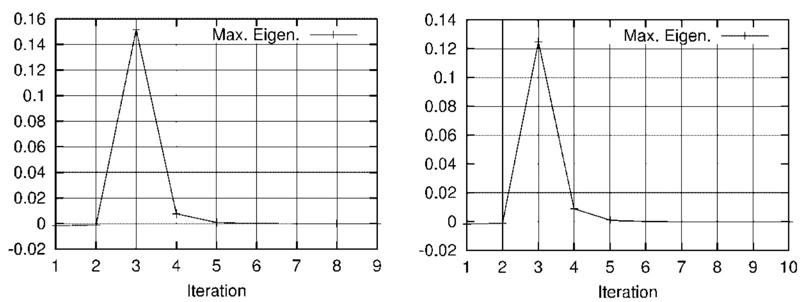
From these results, it can be said that the results of this research are adaptable to the actual problem, which means that it can be suggested to employ the real-part eigenvalue of the system in order to judge the stability of the eco-system.

The critical state is assumed to be obtained by the allowance parameter $\lambda(\lambda_1, \lambda_2, \lambda_3)$ which are multiplied by P, Z, and N in an equilibrium state. The observed data in 1975, 1976, 1991, and 2000 of the lake are employed to identify parameter λ. With the parameter identification



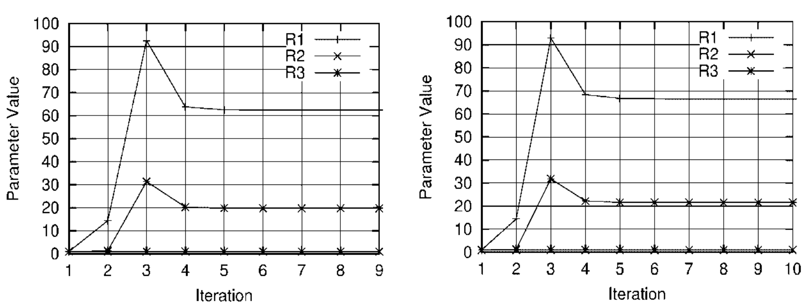
April, 1975 May, 1976

Figure 9. Performance function J.



April, 1975 May, 1976

Figure 10. Maximum eigenvalue.



April, 1975 May, 1976

Figure 11. Parameter values.

technique, critical values of allowance parameter λ_1, λ_2, λ_3 are obtained as the maximum eigenvalue as close as or equal to objective eigenvalues, which are chosen to be all nearly equal to zero.

The results in May 1976 and in May 1976 are shown in Figures 9–11.

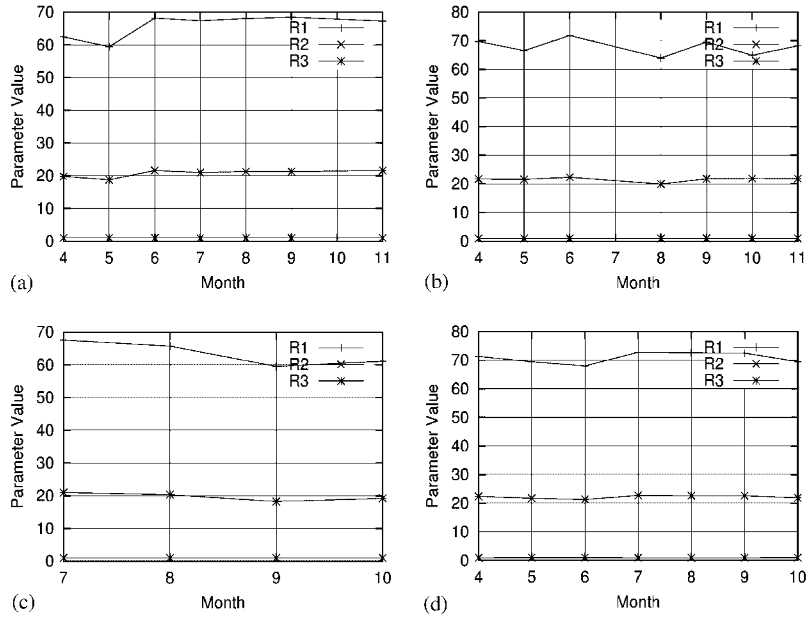


Figure 12. Parameter values in each month. (a) converged λ of 1975, (b) converged λ of 1976, (c) converged λ of 1991 and (d) converged λ of 2000.

In these years, the abnormal multiplication of plankton did not take place. By using parameter λ, the outbreak of plankton in Lake Kasumigaura is estimated. Figure 9 shows the Performance Function J. Figure 10 is the convergence diagram of maximum eigenvalue of the system. Figure 11 represents the convergence of parameter λ. The results in 1975 are converged after the fifth iteration. The data in 1976 are converged after the fifth iteration, too. Objective eigenvalue $\max(R_e\{\sigma\})$ is set as -1.0×10^{-4}. From Figure 11, λ_1, λ_2 and λ_3 in 1975 are converged to 62.37, 0.981 and 19.769, respectively. If these values are multiplied by the original equilibrium distributions, the stability of the system is unstable.
Figure 12 shows convergence values of λ_1, λ_2 and λ_3 in every month.

From each parameter value, it is considered that the system is unstable if the equilibrium concentration of phytoplankton, zooplankton and nutrient are multiplied by 58–72, 0.96–0.98, and 18–23. The sensitivity of zooplankton is shown to be much more significant than those of phytoplankton and nutrient.

7.4. *Discussion of parameter λ*

As one of the discussions of this result, parameter identification technique of one parameter is employed in order to examine the physical meaning of parameter λ.

At first, parameter λ_1 is identified specifying λ_2 and λ_3. Figure 13 shows the computed result. In this case λ_1 is converged to about 68.2 regardless of the value of λ_2 and λ_3. Figure 14 shows the case where λ_2 is identified specifying λ_1 and λ_3. In this case, the converged values differ as the parameter λ_1 changes. Figure 15 shows the convergence of λ_3. Looking at the results in Figure 13, it is shown that the significance of the value λ_1 rules the characteristic

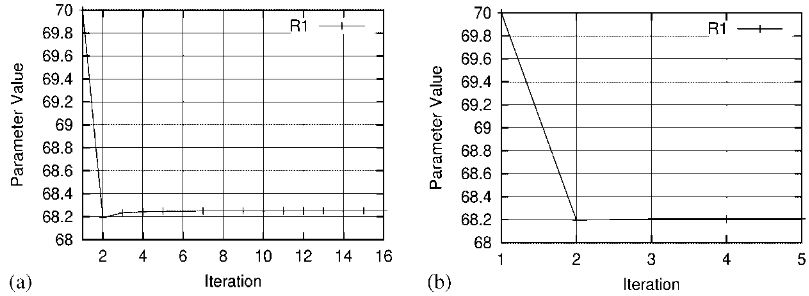


Figure 13. Parameter λ_1: (a) $\lambda_2 = 20.0$, $\lambda_3 = 30.0$ and (b) $\lambda_2 = 1.0$, $\lambda_3 = 40.0$.

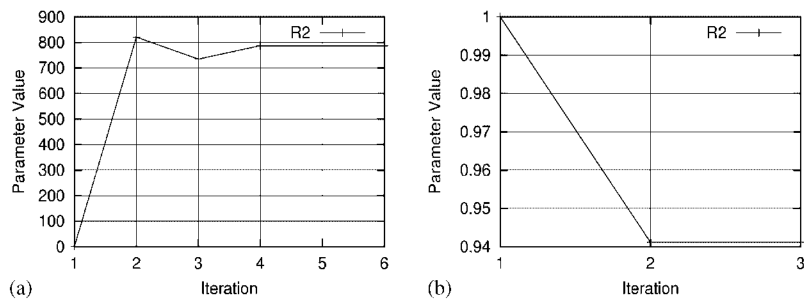


Figure 14. Parameter λ_2: (a) $\lambda_1 = 70.0$, $\lambda_3 = 30.0$ and (b) $\lambda_1 = 68.206$, $\lambda_3 = 40.0$.

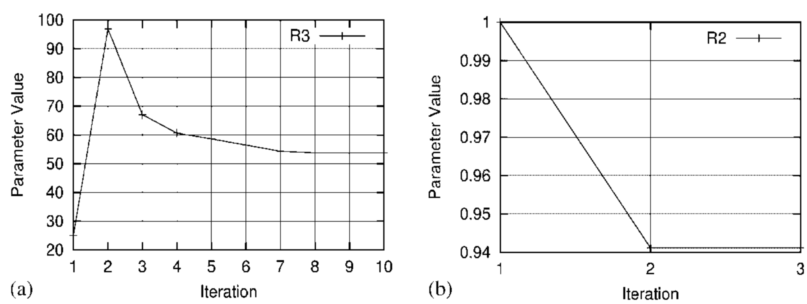


Figure 15. Parameter λ_3: (a) $\lambda_1 = 10.0$, $\lambda_2 = 1.0$ and (b) $\lambda_1 = 68.206$, $\lambda_2 = 1.0$.

of the system. From the above discussions, the influence of the value of λ_1 is dominated in the system because all converged values are varied by the choice of λ_1. Thus it can be said that the concentration of phytoplankton influences the stability of Lake Kasumigaura.

8. CONCLUSIONS

In this research, a method for the estimation of the abnormal multiplication of plankton has been presented. The point of this research is the judging of the system. The approach using an eigenvalue of the system is a relatively new method. Employing an eigenvalue, not only the judgement of the stability of the system but also the prediction of outbreak of plankton can be obtained. In Section 7.3, it is shown that λ_1, λ_2, λ_3 are converged to 58–72, 0.96–0.98, and 18–23. In Section 7.4, it is recognized that the parameter λ_1, which expresses the importance of phytoplankton, rules the system. Changing the value of λ_1, the stability of the system also changes. Especially, the influence of λ_2 on the system is small since the converged value of λ_2 is very different as shown in Figure 14. Actually, the factor of red tide is the outgrowth of phytoplankton. It can be said that the cause of outbreak of plankton is related to the concentration of phytoplankton mathematically.

From this method, prediction of the optimal concentration in the system is possible. It is hoped that the theory of this research will be employed in some other analyses.

REFERENCES

1. Okaichi T. *Science of Red Tide*. Kousei Company Kousei Kaku, 1984.
2. Red Tide occurred at Kanazawa-ku, Yokohama-city: *asahi.com*; 21st, May, 2001.
3. Iwasa Y. *Lakes and Marshes Technology*. Sankai-do, 1990.
4. Matsunashi J. *Fluid Pollution of Environment*. Morikita shuppan, 1993.
5. Ohkubo A. *Ecology and Diffusion*. Tukizi-shuppan, 1984.
6. Franks PJS, Changsheng Chen. Plankton production in tidal fronts: A model of Georges Bank in summer. *Journal of Marine Research* 1996; **54**:631–651.
7. Wroblewskl JS, O'Brien JJ. A spacial model of phytoplankton patchiness. *Marine Biology* 1976; **35**:161–172.
8. Ding Y, Kawahara M. *Bifurcation Analysis of Brown Tide in Tidal Flow Using Finite Element Method.* Kawahara Lab, Chuo-University, 1998.
9. Britton NF. *Reaction-Diffusion Equations and their Application to Biology*. Academic Press: New York, 1999; 109–137.
10. Nayar N, Ortega JM. Computing of selected eigenvalues of generalized eigenvalue problems. *Journal of Computational Physics* 1993; **108**:8–14.

